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An extended channel model for the prediction of motion in elongated homogeneous lakes. Part 1. Theoretical introduction

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Taking account of slenderness of many lakes, a hydrodynamic model is developed. The three-dimensional differential equations are formulated in a curvilinear coordinate system along the 'long' axis of the lake. Applying the method of weighted residuals and expanding the field variables with shape functions over the cross-sections, approximate equations for the fluid motion are derived. The emerging equations form a cross-sectionally discretized set of spatially one-dimensional partial differential equations in the longitudinal lake direction. At first, these channel equations are presented for unspecified fluid properties and arbitrary shape functions, leaving applications possible for inviscid or viscous fluids with arbitrary closure conditions. The channel equations are subsequently specialized for Cauchy series as shape functions. For the free oscillation the simplest channel model is shown to reduce to the classical Chrystal equation. A first-order linear channel model is deduced. It exhibits the essential features of gravitational oscillations in rotating basins, in that it provides wave-type solutions with the characteristics of Kelvin and Poincaré waves. This paper presents the derivation of the equations. Their application to ideal and real basins is deferred to several further papers.

1. Introduction

This paper is concerned with the derivation of an approximate system of equations for slender fluid bodies using free-surface hydrodynamics on the rotating Earth as the illustrating example. This introduction to the theory is complemented by applications in two further papers.

Many relevant theoretical aspects of physical limnology concern the motion of lake water due to wind and pressure fluctuations. These are described mathematically by the Navier–Stokes equations complemented by boundary and initial conditions. Simple models allow construction of analytical solutions to the governing equations; however, most problems, and in particular those for real lakes, need be solved by numerical techniques. These lakes are usually considered as three-dimensional fluid continua. Frequently one or more dimensions are eliminated by an averaging or integration process. Free oscillations of the entire body of the basin are one class of motion. They are known as surface or internal seiches depending on whether the free surface or the thermocline is referred to and are characterized as *gravitational* when accompanied by substantial surface deflections, and as *rotational* or *topographic* when dictated by variations in bathymetry and the rotation of the Earth. Here our focus is on gravitational motions.

The traditional methods of calculating free oscillations are the channel approximations, of which the classical example is the Chrystal equation (Chrystal 1904, 1905). These models only permit a variation in the surface elevation and longitudinal velocity along the channel axis, and velocities transverse to the channel axis are neglected. In rotating basins, the imposition of Kelvin wave dynamics on the channel solution may be employed to construct approximately the positively rotating modes appearing in such basins (see e.g. Defant, 1953). 'Even though the imposition of Kelvin wave dynamics on the channel solution gives very satisfactory results for the lowest mode, its validity for the higher modes breaks down both quantitatively and qualitatively. Kelvin wave hypothesis leads to amphidromic systems, all of which propagate in the counterclockwise direction in the northern hemisphere' (Rao & Schwab 1976). Yet observations by Mortimer & Fee (1976) in Lakes Michigan and Superior indicate that in these basins both positively (counterclockwise) and negatively (clockwise) propagating amphidromic systems are possible. Thus, it is not possible to simulate all slowly rotating surface waves that appear in natural basins with the Kelvin wave hypothesis. Further, the selection of the axis of the lake is rather arbitrary, and as Hamblin (1972) remarks, '...there is an element of subjectivity in prescribing the channel axis which can cause error...'.

It is clear then that a satisfactory treatment of seiches in an arbitrary basin requires an attack on the two-dimensional (and when including viscosity effects three-dimensional) problem. Such models (Hamblin 1972; Rao & Schwab 1976) are based on particular forms of the tidal operator; the discretization makes explicit use of the two-dimensionality of the region of solution for which the classical channel equations break down because the basins are not elongated. On the other hand, some two-dimensional models also have disadvantages in that they may not allow a satisfactory resolution across the smaller dimension of a slender lake, and may even fail in modelling wave propagation in long curved channels (see Rodenhuis 1980). For instance for small lakes numerical stability criteria may force us to use finite-difference mesh sizes so large as to virtually destroy the advantages of the two-dimensional model over the one-dimensional one. And in a bend of a narrow basin the shoreline may be so poorly approximated that false reflections prevent a wave from travelling around the bend.

Here, we develop an extended channel model for *curved, elongated rotating basins*; which not only accounts for the curvature of the axis but also simulates the characteristic behaviour of waves in rotating basins without excessive computational effort. The aim is thereby twofold. Firstly, a systematic, rational procedure is sought by which a series of *channel models* is obtained. For free oscillations the emerging channel theory should improve on the classical Chrystal equation, allowing for positive and negative amphidromic systems. Secondly, the extended models are regarded as *approximations* by which the spatially two- or three-dimensional equations of fluid motion are replaced by a one-dimensional set of equations in order to predict the fluid motion with reasonable accuracy. Here the *minimum effort* is sought that *suffices* to predict the motion of the original problem with reasonable accuracy.

Prominent examples of one-dimensional models derived from three-dimensional continuous media are the theories of rods and jets (see e.g. Antman 1972; Doekmeci 1972; Green, Laws & Naghdi 1974; Green, Naghdi & Wenner 1974*a*,*b*; Green & Naghdi 1976; Naghdi 1979). The methods of derivation are diverse, and authors often emphasize the formal connection of the reduced models with theories of Cosserat curves, but the methods can essentially be interpreted as particular applications of the principle of weighted residuals (Finlayson 1972). They are seldom regarded as a step towards a numerical scheme akin to that of Kantorovich, or the method of lines (Kerr 1967; Aktas, 1979), although this angle of interpretation opens interesting new insight.

In this paper we outline the derivation of the channel equations from the three-dimensional field equations and boundary conditions of fluid mechanics of a rotating system. The selection of a curvilinear co-ordinate system that has as principal co-ordinate an axis of the channel-like lake allows development of such a channel model and reduces the 'element of subjectivity' in prescribing the channel axis. This major axis is complemented by two co-ordinates within the cross-section selected to be perpendicular to the major axis. The channel model is derived using the method of weighted residuals. The technique is to prescribe the shape of the channel-model solution in the cross-sections as truncated sets of functions leaving variation of the solution free with respect to time and the longitudinal co-ordinate. The integrals appearing in the weighted residual expressions can therefore be performed explicitly in the cross-section, so that the three-dimensional partial differential equations are transformed to a set of partial differential equations in time and in the longitudinal co-ordinate. This cross-sectional integration is not entirely straightforward, however, as in the original problem there arise boundary conditions at the bottom and free surface, which must be taken into account when formally reducing the system from a spatially three-dimensional to a one-dimensional one. A hierarchy of models is thus established according to the number of shape functions selected to represent each variable. In view of future applications viscous-drag and turbulent-friction effects are included in the formulation. Nonlinear terms are kept, in general, but free-surface amplitudes are assumed to be small in the sense that cross-sections are constant and evaluated at their equilibrium state. This is only a matter of convenience, and the assumption can easily be lifted. Equations are then specialized for an inviscid fluid and results are discussed for these. In particular, the simplest model reduces to the Chrystal equation, and the first-order model reflects already the essential features of long gravitational waves. Applications of the method to other physical systems are hinted, but the demonstration that 'it works' for rotating fluids is deferred to three other forthcoming articles (Raggio & Hutter 1982*a*, *b*; Hutter & Raggio, 1982).

2. Governing equations

We are concerned here with the dynamics of an incompressible fluid with free surface in a steady rotating basin. Let Ω denote the domain filled by the fluid and $\partial\Omega$ its boundary consisting of the free surface $\partial\Omega_\sigma$ and the bottom boundary $\partial\Omega_n$. To describe the motion of fluid particles in Ω, a right handed plane curvilinear orthogonal co-ordinate system is introduced (see figure 1). In a first step the lake axis is selected within the undisturbed lake surface and, then, complemented by two other axes, one horizontal and the other vertical. The curve parameter on the axis is denoted by s, the co-ordinate measured horizontally by n and that on the vertical axis by z, which is positive upwards. The selection of the (s,n,z)-system is subjective, but for each individual lake the lake axis may suggest itself in a natural fashion.

In subsequent developments the prescribed lake axis plays a prominent role. Its equation $\mathbf{x} = \tilde{\mathbf{x}}(s)$ and its curvature $K(s)$ are assumed to be given functions of arc

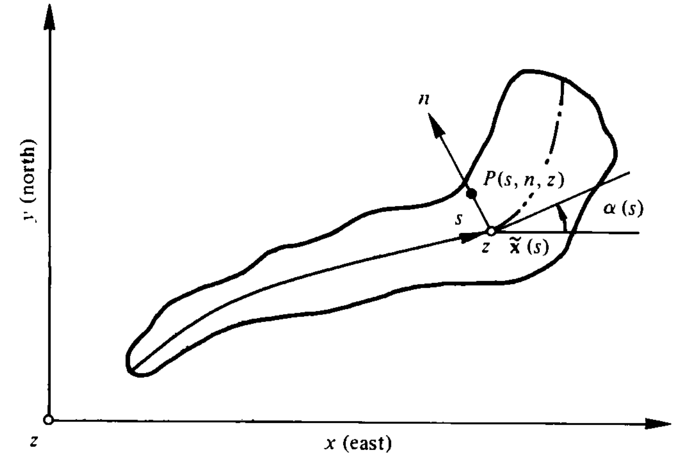

FIGURE 1. Definition of the (s, n, z)-co-ordinate system.

length s. Of importance is also the Jacobian of the metric tensor of the orthogonal system (s, n, z); it is given by

$$J(s,n) = 1 - K(s)\,n, \tag{2.1}$$

and must be positive if a point in Ω is to be described uniquely. This is the only coercive condition that may restrict the location of the axis.

Balance of mass and momentum are the basic laws in fluid dynamics describing physical processes in a purely mechanical model. In the curvilinear coordinate system introduced above and for an *incompressible* liquid these equations assume the component form

$$\frac{1}{J}\frac{\partial v_s}{\partial s}+\frac{\partial v_s}{\partial n}+\frac{\partial v_s}{\partial z}-\frac{K}{J}v_n = 0, \tag{2.2}$$

$$\rho\left[\frac{\partial v_s}{\partial t}+\frac{v_s}{J}\frac{\partial v_s}{\partial s}+v_n\frac{\partial v_s}{\partial n}+v_z\frac{\partial v_s}{\partial z}-\frac{K}{J}v_s v_n\right] + \frac{1}{J}\frac{\partial p}{\partial s}-\rho f v_n-\left(\frac{1}{J}\frac{\partial T^{\mathrm{E}}_{ss}}{\partial s}+\frac{\partial T^{\mathrm{E}}_{sn}}{\partial n}+\frac{\partial T^{\mathrm{E}}_{sz}}{\partial z}\right)+\frac{K}{J}\left(\frac{K'n^2}{J}T^{\mathrm{E}}_{ss}+2T^{\mathrm{E}}_{sn}\right) = 0, \tag{2.3a}$$

$$\rho\left[\frac{\partial v_n}{\partial t}+\frac{v_s}{J}\frac{\partial v_n}{\partial s}+v_n\frac{\partial v_n}{\partial n}+v_z\frac{\partial v_n}{\partial z}+\frac{K}{J}v_s^2\right] + \frac{\partial p}{\partial n}+\rho f v_s-\left(\frac{1}{J}\frac{\partial T^{\mathrm{E}}_{sn}}{\partial s}+\frac{\partial T^{\mathrm{E}}_{nn}}{\partial n}+\frac{\partial T^{\mathrm{E}}_{nz}}{\partial z}\right)+\frac{K}{J}\left(J^2T^{\mathrm{E}}_{ss}-T^{\mathrm{E}}_{nn}\right) = 0, \tag{2.3b}$$

$$\rho\left[\frac{\partial v_z}{\partial t}+\frac{v_s}{J}\frac{\partial v_z}{\partial s}+v_n\frac{\partial v_z}{\partial n}+v_z\frac{\partial v_z}{\partial z}\right] + \frac{\partial p}{\partial z}+\rho g-\left(\frac{1}{J}\frac{\partial T^{\mathrm{E}}_{sz}}{\partial s}+\frac{\partial T^{\mathrm{E}}_{nz}}{\partial n}+\frac{\partial T^{\mathrm{E}}_{zz}}{\partial z}\right)+\frac{K}{J}T^{\mathrm{E}}_{nz} = 0. \tag{2.3c}$$

Here, ρ is the density, $\mathbf{v}$ velocity, $\mathbf{T}^{\mathrm{E}}$ Cauchy stress deviator, p pressure and g acceleration due to gravity. Furthermore, $K' = dK/ds$, and $f = 2\,|\boldsymbol{\Omega}|\sin\phi$ is the Coriolis parameter, which depends on the rotation of the Earth $|\boldsymbol{\Omega}|$ and on latitude ϕ, a possible z-component being absorbed in g (see Krauss 1973). It should also be mentioned that the components of $\mathbf{v}$ and $\mathbf{T}^{\mathrm{E}}$ are the *physical* components referred to a basis of unit vectors along the (s,n,z)-directions. A derivation of (2.2) and (2.3) from first principles is given by Raggio (1981).

Equations (2.2), (2.3) must be complemented by phenomenological statements regarding extra stress $\mathbf{T}^{\mathrm{E}}$ which in turbulent motion can be identified with the Reynolds stresses, if, as is usually the case, molecular viscous effects are ignored. There are numerous turbulent closure conditions, one using the analogy with the molecular viscosity and others using transport equations for the stresses (see Launder & Spalding 1972; Bradshaw 1976); at this stage the precise formulation of these phenomenological assumptions is left unspecified since this will not affect the subsequent calculations.

To solve the field equations, *boundary* and *initial conditions* must be prescribed. As to the former, kinematic and dynamic conditions apply. Let

$$F \equiv \begin{cases} \hat{F}_{\xi}(s,n,z,t) = \xi(s,n,t) - z \equiv 0 \quad \text{on} \quad \partial\Omega_{\sigma}, \\ F_H(s,n,z) = H(s,n) - z \equiv 0, \quad \text{on} \quad \partial\Omega_n, \end{cases} \tag{2.4}$$

be the equations defining the free and bottom surface. Here and henceforth hats indicate functions which are defined on the free surface only. With wind-stresses t_s^*, t_n^* and atmospheric pressure p_{at}, we may thus write as kinematic and dynamic conditions on $\partial\Omega_{\sigma}$

$$\left.\begin{aligned} &\frac{\partial\xi}{\partial t} + \frac{1}{J}\frac{\partial\xi}{\partial s}v_s + \frac{\partial\xi}{\partial n}v_n - v_r = 0, && (2.5)\\ &\frac{1}{J}(-p + T^{\mathrm{E}}_{ss})\frac{\partial\xi}{\partial s} + T^{\mathrm{E}}_{sn}\frac{\partial\xi}{\partial n} - T^{\mathrm{E}}_{sz} = -t_s^* l_{\xi}, && (2.6a)\\ &\frac{1}{J}T^{\mathrm{E}}_{sn}\frac{\partial\xi}{\partial s} + (-p + T^{\mathrm{E}}_{nn})\frac{\partial\xi}{\partial n} - T^{\mathrm{E}}_{nz} = -t_n^* l_{\xi}, && (2.6b)\\ &\frac{1}{J}T^{\mathrm{E}}_{sz}\frac{\partial\xi}{\partial s} + T^{\mathrm{E}}_{zn}\frac{\partial\xi}{\partial n} - (-p + T^{\mathrm{E}}_{zz}) = +p^*_{\mathrm{at}} l_{\xi}, && (2.6c) \end{aligned}\right\} \quad \text{at} \quad z = \xi(s,n,t),$$

with

$$l_{\xi} \equiv \left[1 + \frac{1}{J^2}\left(\frac{\partial\xi}{\partial s}\right)^2 + \left(\frac{\partial\xi}{\partial n}\right)^2\right]^{\frac{1}{2}} = 1 + O(\xi^2).$$

Equation (2.5) expresses the fact that $\partial\Omega_{\sigma}$ is material, and (2.6*a–c*) are the three components of the continuity of surface traction.

At the bottom boundary the kinematic condition (2.4) and a viscous sliding law imply the relations

$$\left.\begin{aligned} &\frac{1}{J}\frac{\partial H}{\partial s}v_s + \frac{\partial H}{\partial n}v_n - v_z = 0, && (2.7)\\ &v_s = -\frac{R}{l_H}\left\{\frac{1}{J}T^*_{ss}\frac{\partial H}{\partial s} + T^*_{sn}\frac{\partial H}{\partial n} - T^*_{sz}\right\}, && (2.8a)\\ &v_n = -\frac{R}{l_H}\left\{\frac{1}{J}T^*_{sn}\frac{\partial H}{\partial s} + T^*_{nn}\frac{\partial H}{\partial n} - T^*_{nz}\right\}, && (2.8b)\\ &v_z = -\frac{R}{l_H}\left\{\frac{1}{J}T^*_{sz}\frac{\partial H}{\partial s} + T^*_{zn}\frac{\partial H}{\partial n} - T^*_{zz}\right\}, && (2.8c) \end{aligned}\right\} \quad \text{at} \quad z = H(s,n).$$

in which

$$\mathbf{T}^* = \mathbf{T}^{\mathrm{E}} - \mathbf{n}\cdot\mathbf{T}^{\mathrm{E}}\cdot\mathbf{n}, \quad l_H = \left[1 + \frac{1}{J^2}\left(\frac{\partial H}{\partial s}\right)^2 + \left(\frac{\partial H}{\partial n}\right)^2\right]^{\frac{1}{2}}. \tag{2.8d, e}$$

Here $\mathbf{n}$ is the unit exterior normal vector, and R is a friction coefficient, which is not necessarily constant, but may be a function of velocity, bed roughness etc. The boundary condition (2.8) includes with $R = 0$ the no-slip condition and automatically satisfies the kinematic condition $\mathbf{v}\cdot\mathbf{n} = 0$. $R \to \infty$ necessarily requires $\mathbf{T}^{\mathrm{E}} \equiv \mathbf{0}$, so a viscous bottom sliding law can consequently only be introduced if the extra stress tensor does not identically vanish. A similar remark also holds for wind stress, as can be seen from (2.6). Since our ultimate aim is in wind-induced currents (storm surges) we shall keep the viscous terms here, even though the first applications of the theory will only be on inviscid fluid flow.

We close this section by stating *initial conditions* in the form

$$\mathbf{v} = \mathbf{v}^{**}(s, n, z), \quad \xi = \xi^{**}(s, n). \tag{2.9}$$

The functions $\mathbf{v}^{**}(\;)$ and $\xi^{**}(\;)$ are not completely arbitrary, as they should conform with the balance laws and boundary conditions. For instance, v^{**} must satisfy the kinematic bottom boundary condition, and should also satisfy the divergence condition $\operatorname{div}(\mathbf{v}^{**}) = 0$. A common initial condition satisfying these restrictions is for example a start from rest, $\mathbf{v}^{**} = \mathbf{0}$, $\xi^{**} = 0$.

The above equations comprise a complete formulation of the three-dimensional initial-boundary-value problem of the motion of a fluid with free surface in a basin that rotates with steady angular velocity. The equations are limited to processes in which temperature is a passive quantity, and, strictly speaking, they can only be solved once a closure condition for the Reynolds stresses is presented. For an application in limnology the third momentum equation is usually further simplified by invoking the *hydrostatic-pressure assumption.* This reduces (2.3*c*) to

$$p = \rho g(\xi(s, n, t) - z). \tag{2.10}$$

The following developments will be restricted to this case.

3. The method of weighted residuals

The purpose of a one-dimensional model which replaces the three-dimensional continuum is to obtain a computationally more easily accessible description which provides better insight into the physical system under consideration. To derive such one-dimensional formulations one uses either asymptotic expansions (compare derivations of the Korteweg–de Vries equation and its variants) or starts from generalized (Cosserat) continua or transforms the three-dimensional equations to a weak (integral) form. It is probably fair to say that the latter two methods are embodied in the more general principle of weighted residuals (see Finlayson 1972). Here we shall use this well-known principle, which mathematically corresponds to the projection method (Kantorovich & Krylov 1958), but its use in our application is novel and thus requires a brief exposition of the method.

The essential idea of the method of weighted residuals is as follows. Let $\mathbf{R}_{\Omega} = \mathbf{0}$ be the set of field equations defined over Ω and $\mathbf{R}_{\partial\Omega_\alpha} = \mathbf{0}$ ($\alpha = 1, 2, \ldots, \nu$) the associated boundary conditions, of which each holds at the part $\partial\Omega_\alpha$ of the total boundary. In the context of §2, $\mathbf{R}_{\Omega} = \mathbf{0}$ stands for (2.2) and (2.3) (written here in vectorial form); further, $\nu = 2$ with $\partial\Omega_1 = \partial\Omega_\sigma$, $\partial\Omega_2 = \partial\Omega_n$, and $\mathbf{R}_{\partial\Omega_\alpha} = \mathbf{0}$ represents (2.5)–(2.8). (The numbers of components in $\mathbf{R}_{\Omega}$ and $\mathbf{R}_{\partial\Omega_\alpha}$ need not be the same, and the domains of their definition are different, which explains the use of the subscripts). Now form the scalar product of $\mathbf{R}_\Omega = \mathbf{0}$ and $\mathbf{R}_{\partial\Omega_\alpha} = \mathbf{0}$ with arbitrary weighting functions $\delta\mathbf{w}_\Omega$ and $\delta\mathbf{w}_{\partial\Omega_\alpha}$, and integrate the resulting scalars over those volume or surface parts, on which the respective equations hold. Upon addition of the resulting expressions one obtains

$$\delta I = \int_\Omega \mathbf{R}_\Omega \cdot \delta\mathbf{w}_\Omega \, d\Omega + \sum_\alpha \int_{\partial\Omega_\alpha} \mathbf{R}_{\partial\Omega_\alpha} \cdot \delta\mathbf{w}_{\partial\Omega_\alpha} \, d\,\partial\Omega_\alpha = 0 \tag{3.1}$$

as the weak form of the original initial-boundary-value problem. Here we have used the δ-symbol as a reminder that the weighting functions are arbitrary; the quantity I by itself is not defined, in general, unless the equations are self-adjoint. A solution of the boundary-value problem always implies $\delta I = 0$; that the converse is true for arbitrary weighting functions follows from a fundamental lemma of the calculus of variations.

Equation (3.1) is the basis for the approximation to deduce the spatially one-dimensional model. To this end all weighting functions $\delta\mathbf{w}$ and each unknown field variable (velocity components, surface elevation etc.); say x for brevity, are postulated as products of two truncated sets in the form

$$x = \sum_{i=1}^{N} \phi_i x_i = \boldsymbol{\phi}^{\mathrm{T}} . \mathbf{x}, \quad \delta\mathbf{w} = \sum_{i=1}^{N} \psi_i \delta\mathbf{w}_i = \boldsymbol{\psi}^{\mathrm{T}} . \delta\mathbf{w}, \tag{3.2}$$

where the ϕ_i and ψ_i generate sets of linearly independent known functions, called shape, basis or trial functions, and the x_i and $\delta\mathbf{w}_i$ constitute sets of unknown (x_i) and arbitrary $(\delta\mathbf{w}_i)$ functions. The $\boldsymbol{\phi}$- and $\boldsymbol{\psi}$-functions are chosen to depend on some of the independent variables, and the $\mathbf{x}$ and $\delta\mathbf{w}$ depend on the remaining variables. This amounts to separation of the variables into known sets $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ and unknown sets $\mathbf{x}$ and $\delta\mathbf{w}$. To be more specific,

$$\left.\begin{aligned} x(s,n,z,t) &= \boldsymbol{\phi}^{\mathrm{T}}(n,z) . \mathbf{x}(s,t), \\ \delta\mathbf{w}(s,n,z,t) &= \boldsymbol{\psi}^{\mathrm{T}}(n,z) . \delta\mathbf{w}(s,t), \end{aligned}\right\} \tag{3.3}$$

in which $\boldsymbol{\phi}$ (and $\boldsymbol{\psi}$) describe the *distribution* of x (or $\delta\mathbf{w}$) over cross-sections of the lake. Restriction of (3.2) to $N = 1$ often amounts to the construction of *similarity solutions*, and N (not necessarily $= 1$) defines the order of the model. The sets $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ may be constructed from products of polynomials, or other appropriate functions. In principle they could also vary with s without violation of the general developments, but here such a dependence will be omitted. Further, one may choose $\boldsymbol{\phi} = \boldsymbol{\psi}$, which we will do later on; this is again special and corresponds to a *Galerkin procedure*.

By introducing the representations (3.3) into (3.1), integrations over Ω and $\partial\Omega_\alpha$ can be split into cross-sectional integrals over the co-ordinates n and z, followed by an integration along the axis. Because $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are known, integrations over n and z can be performed explicitly. Structurally, (3.1) thus has the form

$$\delta I = \int_{s_1}^{s_2} \langle \mathbf{A}(\mathbf{x}), \quad \delta\mathbf{w} \rangle \, ds = 0, \tag{3.4}$$

in which $\langle\,,\rangle$ is a bilinear functional, and integration is along the lake axis from $s = s_1$ to $s = s_2$. Since $\delta\mathbf{w}$ is arbitrary (3.4) implies (using the lemma of the calculus of variations) $\mathbf{A}(\mathbf{x}(s,t)) = \mathbf{0}$, which is the approximate set of spatially one-dimensional equations. In this process of reduction certain volume integrals may be transformed to surface integrals by using Green's theorem, but physical arguments must suggest which of the global representations should be regarded as the correct one. We shall examine this difficulty further.

The discussion above outlines the general procedure that must be followed in the method of weighted residuals. All that is needed is to demonstrate its application explicitly; the field equations (2.2), (2.3) and boundary conditions (2.5), (2.6) and (2.8) have already been identified. These equations will be multiplied by the weighting functions and then integrated over their domains of definition. With the weighting functions $\delta\mathbf{v}_1$, $\delta\lambda_1$ defined over Ω; $\delta\mathbf{v}_2$, $\delta\lambda_2$ defined over $\partial\Omega_\sigma$, and $\delta v_3, \delta\lambda_3$ defined over $\partial\Omega_n$ we obtain

$$\sum_{\alpha=1}^{6} \delta I_\alpha = 0, \tag{3.5}$$

with

$$\delta I_1 = \int_{t_1}^{t_2} dt \iiint_\Omega \operatorname{div} \mathbf{v}\, \delta\lambda_1\, dV, \tag{3.6a}$$

$$\delta I_2 = \int_{t_1}^{t_2} dt \iiint_\Omega \left\{\rho \frac{dv}{dt} + \operatorname{grad} p - \operatorname{div} \mathbf{T}^{\mathrm{E}} - \rho \mathbf{f}\right\} . \delta\mathbf{v}_1\, dV, \tag{3.6b}$$

$$\delta I_3 = \int_{t_1}^{t_2} dt \iint_{\partial\Omega_\sigma} \left\{\frac{\partial F_\xi}{\partial t} + \operatorname{grad} F_\xi . \mathbf{v}\right\} \delta\lambda_2\, da, \tag{3.6c}$$

$$\delta I_4 = \int_{t_1}^{t_2} dt \iint_{\partial\Omega_\sigma} \{(-p\mathbf{1} + \mathbf{T}^{\mathrm{E}})\,\mathbf{n} - \mathbf{t}^*\} . \delta\mathbf{v}_2\, da, \tag{3.6d}$$

$$\delta I_5 = \int_{t_1}^{t_2} dt \iint_{\partial\Omega_n} \frac{\operatorname{grad} F_H}{|\operatorname{grad} F_H|} . \mathbf{v}\, \delta\lambda_3\, da, \tag{3.6e}$$

$$\delta I_6 = \int_{t_1}^{t_2} dt \iint_{\partial\Omega_n} \left\{\mathbf{T}^*\mathbf{n} + \frac{1}{R}\mathbf{v}\right\} . \delta\mathbf{v}_3\, da, \tag{3.6f}$$

where dV is a volume element in the domain Ω, and da a surface element on the surface $\partial\Omega$. Integrations over time from $t_1 \leqslant t \leqslant t_2$ arise because our physical problem is an *initial*-boundary-value problem. Conceptually, these time integrations are important. In actual calculations they do not play a role, however, and will henceforth be omitted. Equations (3.6*a*, *b*) are the residuals corresponding to the field equations (2.2) and (2.3); (3.6*c*, *d*) correspond to the free-surface boundary conditions and (3.6*e*, *f*) to the bottom boundary conditions.

The major goal in § 4 is to explain how (3.5) and (3.6) are used to derive a spatially one-dimensional model of water motion in rotating basins.

4. Derivation of an approximate channel model for barotropic motions in a lake

With the curvilinear co-ordinate system introduced in § 2 a preference for the long direction of the lake is naturally built into the governing equations, which can now be reduced to one-dimensional form by discretizing them in the cross sectional co-ordinates (n and z of § 2). The basis for this reduction is the fact that variations in the long direction are more important than the cross-sectional variations suggesting the shape-function expansions (3.3), in which the dimensions of $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ are chosen to coincide (same value for N in (3.2)), to yield a determinate system. Although each variable could have its own representation (3.3), the ensuing developments are based on the more restricted expansions for the velocity components and surface elevation

$$(v_s, v_n, v_z) = \boldsymbol{\phi}^{\mathrm{T}} . (\mathbf{v}_s, \mathbf{v}_n, \mathbf{v}_z), \quad \xi = \hat{\boldsymbol{\phi}}^{\mathrm{T}} . \boldsymbol{\xi}, \tag{4.1}$$

in which $\boldsymbol{\phi} = \boldsymbol{\phi}(n, z)$, $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}(n)$; $\boldsymbol{\phi}^{\mathrm{T}}$ denoting the transpose of $\boldsymbol{\phi}$. The weighting functions δv_s, δv_n, δv_z and λ introduced in (3.6) are similarly expanded:

$$\left.\begin{aligned} (\delta v_s, \delta v_n, \delta v_z, \delta \lambda_1) &= \boldsymbol{\psi}^{\mathrm{T}} . (\delta \mathbf{v}_s, \delta \mathbf{v}_n, \delta \mathbf{v}_z, \delta \boldsymbol{\lambda}_1), \\ (\delta \lambda_2, \delta \lambda_3) &= \hat{\boldsymbol{\psi}}^{\mathrm{T}} . (\delta \boldsymbol{\lambda}_2 \delta \boldsymbol{\lambda}_3), \end{aligned}\right\} \tag{4.2}$$

with $\boldsymbol{\psi} = \boldsymbol{\psi}(n, z)$ and $\hat{\boldsymbol{\psi}} = \hat{\boldsymbol{\psi}}(n)$.

4.1. *Equations of motion*

In this subsection the equations of motion for a lake are presented using the previous expansions (4.1) and (4.2) and the principle of weighted residuals introduced in §3. Calculations are frequently very involved. In order not to sidetrack the main ideas, auxiliary calculations and space-filling definitions are appended at the end, or only key steps are indicated.

We begin by noticing that one possible set of weighting functions in (3.5) and (3.6) is such that $\Sigma_{a=1}^{8} \delta I_a = 0$ reduces to $\delta I_1 = 0$. By way of illustration we make this choice and obtain

$$\begin{aligned} \delta I_1 &= \int_{t_1}^{t_2} dt \iiint_{\Omega} \operatorname{div} \mathbf{v}\, \delta\lambda_1\, dV = \int_{t_1}^{t_2} dt \int_{s_0}^{s_1} ds \iint_Q \operatorname{div} \mathbf{v}\, \delta\lambda_1\, J\, dn\, dz \\ &= \int_{t_1}^{t_2} dt \int_{s_0}^{s_1} ds \iint_Q \delta\boldsymbol{\lambda}_1^{\mathrm{T}} . \boldsymbol{\psi} \left\{ \frac{1}{J}\frac{\partial v_s}{\partial s} + \frac{\partial v_n}{\partial n} + \frac{\partial v_z}{\partial z} - \frac{K}{J} v_n \right\} J\, dn\, dz \\ &= \int_{t_1}^{t_2} dt \int_{s_0}^{s_1} ds\, \delta\boldsymbol{\lambda}_1^{\mathrm{T}} . \left\{ \iint_Q \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn\, dz \frac{\partial \mathbf{v}_s}{\partial s} \right. \\ &\quad + \left[\iint_Q \boldsymbol{\psi} \otimes \boldsymbol{\phi}_{,n} J\, dn\, dz - K \iint_Q \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn\, dz \right] \mathbf{v}_n \\ &\quad \left. + \iint_Q \boldsymbol{\psi} \otimes \boldsymbol{\phi}_{,z} J\, dn\, dz\, \mathbf{v}_z \right\} = 0, \end{aligned} \tag{4.3}$$

in which $\otimes$ is the exterior product and $\boldsymbol{\phi}_{,n} \equiv \partial\boldsymbol{\phi}/\partial n$ and $\boldsymbol{\phi}_{,z} \equiv \partial\boldsymbol{\phi}/\partial z$. (Note that the symbolic notation in the various steps of (4.3) is used on different vector spaces. In the first line this vector space is the physical space, whereas in the following three lines it is the shape-function space. In subsequent developments symbolic notation will pertain to shape functions, and variables in physical 3-space will be written in indical notation.) Evidently the volume integral has first been split into an integration over the cross-section Q, followed by an integration over arc length along the axis. Steps two and three then consist of a substitution of the expansions (4.1) and (4.2). Clearly, the double integrals over the cross-section can be performed explicitly as the integrand functions only depend on the bathymetry of the basin, and are known once and for all if the basin geometry is prescribed. Hence, in short (4.3) becomes

$$\delta I_1 = \int_{t_1}^{t_2} dt \int_{s_0}^{s_1} ds\, \delta\boldsymbol{\lambda}_1^{\mathrm{T}} . \left\{ \mathbf{C}^{(0)} \frac{\partial \mathbf{v}_s}{\partial s} + (\mathbf{C}_{\phi n}^{(1)} - K\mathbf{C}^{(0)})\, \mathbf{v}_n - \mathbf{C}_{\phi z}^{(1)} \mathbf{v}_z \right\} = 0, \tag{4.4}$$

in which the cross-sectional coefficients $\mathbf{C}^{(m)}$, $\mathbf{C}^{(m)}_{\phi_n}$, $\mathbf{C}^{(m)}_{\phi_z}$ are listed explicitly in appendix A and can be treated as known functions of s.† Invoking the lemma of the calculus of variations, since $\delta\boldsymbol{\lambda}_1$ is arbitrary, implies

$$\left.\mathbf{C}^{(0)}\frac{\partial \mathbf{v}_s}{\partial s}+(\mathbf{C}^{(1)}_{\phi_n}-K\mathbf{C}^{(0)})\,\mathbf{v}_n-\mathbf{C}^{(1)}_{\phi_z}\mathbf{v}_z\right\} = 0. \tag{4.5}$$

This is a set of N differential equations in the spatial variable s for the N-vectors $\mathbf{v}_s, \mathbf{v}_n, \mathbf{v}_z$. Clearly, since (4.5) derives from the continuity equation, it may be regarded as a set of equations approximating three-dimensional mass balance. Choosing other weighting functions one can proceed in this fashion and set $\Sigma_\alpha \delta I_\alpha = 0$ to produce further equations until a determinate set of differential equations is obtained from which $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$ and $\boldsymbol{\xi}$ can be calculated. However, it is found that a spatially one-dimensional theory using (4.5) as the appropriate mass-balance equation is not capable of predicting long gravity waves. The difficulty is a subtle one and warrants closer attention.

We may start our explanation by way of analogy with shock waves. Smooth shock-free solutions are obtained from differentiable 'local' or point forms of the physical-balance laws, but shock conditions follow from the integrated 'global' forms of these laws. It is these integrated laws which describe the physics, and the local laws are deduced from them by transforming (through the use of Green's theorem) boundary integrals into volume integrals. It is not difficult to anticipate from this that the correct channel equations will be obtained if mass and momentum balances are established for cross-sectional averages. These laws are global in the sense that integrations over cross-sections are performed and boundary conditions along the bounding bottom and the surface lines are incorporated. With regard to the application of the principle of weighted residuals this means that the functionals δI_α $(\alpha = 1, 2, \ldots, 6)$ in the basic statement (3.6) must be combined and appropriate integrations by parts be performed such that the cross-sectionally averaged mass- and momentum-balance laws can be identified.

A simple and familiar example illustrating the above arguments is the derivation of the *kinematic wave equation*. It combines the continuity equation and the kinematic boundary conditions at the top and bottom surfaces, for plane flow,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0,$$

$$\frac{\partial \xi}{\partial t}+\frac{\partial \xi}{\partial x}u-v = 0 \quad \text{at} \quad y = \xi(x,t),$$

$$\frac{\partial H}{\partial x}u+v = 0 \quad \text{at} \quad y = -H(x),$$

† The sub- and superscripts in the matrices $\mathbf{C}$ (and later in other quantities) have suggestive meanings which are evident from a comparison of (4.3) and (4.4). A superscript in parentheses will always mean that in the defining equation the Jacobian J of the curvilinear co-ordinate system appears as a weighting factor with a power indicated by the superscript. A subscript indicates that the integral function appearing in the definition of the respective matrix integral involves differentiations with respect to the indicated variable, etc. A hat, see later equations, will indicate that shape functions are involved which are only defined on the free surface. We have found this, perhaps cumbersome, notation to be the simplest one.

by integrating the first between $y = -H$ and $y = \xi$, resulting in

$$\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial x}\int_{-H}^{\xi} u\,dy = 0,$$

which is the global depth-integrated mass balance. If ξ and $\int_{-H}^{\xi} u\,dy$ are functionally related this equation becomes the kinematic wave equation.

A cross-sectional average of the balance laws of mass and momentum akin to the above kinematic wave equation has to be set in evidence in the application of the principle of weighted residuals. The 'correct' mass balance statement emerges when the combination $\delta I_1+\delta I_3+\delta I_5$ is considered (these terms represent the continuity equation and the kinematic boundary conditions). Inspection of (3.5) and (3.6) shows that the weighting functions can be selected so that this sum vanishes. This expression is now transformed in the same way that the kinematic wave equation in plane flow was deduced, but the process is complicated because of the complex geometry of the slender body. In order not to direct the reader from the main ideas, this calculation is performed explicitly in appendix B. Here we simply state the result. Accordingly, $\delta I_1+\delta I_3+\delta I_5 = 0$ implies

$$\hat{\mathbf{Z}}^{(1)}\frac{\partial \boldsymbol{\xi}}{\partial t}+\frac{\partial}{\partial s}(\mathbf{C}^{(0)}\mathbf{v}_s)-\mathbf{C}^{(1)}_{\psi_n}\mathbf{v}_n-\mathbf{C}^{(1)}_{\psi_z}v_z = \mathbf{0}, \tag{4.6}$$

where the coefficient matrices are defined in appendix A. This equation indeed resembles the structure of the kinematic wave equation (see Whitham, 1974), but unlike the latter it is a statement involving vectors. When variables are expanded in terms of a single constant shape function ($N = 1$ in (3.2)] (4.6) becomes identical with the classical kinematic wave equation.

In much the same way we proceed with the derivation of the global one-dimensional version of the momentum equation. The dynamic expression δI_2 of (3.6*b*) is combined with the corresponding boundary functionals (involving stresses) δI_4 and δI_6, and the variational equation,

$$\delta I_2+\delta I_4+\delta I_6 = 0, \tag{4.7}$$

using integration by parts is so transformed that the emerging functional can be interpreted as the weighted integral of a global momentum balance of a one-dimensional channelized system. The detailed calculations are very lengthy and tedious, therefore the details will be omitted. The result of (4.7) is

$$\begin{aligned}\rho_0\left\{\mathbf{C}^{(1)}\frac{\partial \mathbf{v}_s}{\partial t}+\boldsymbol{\mathscr{E}}^{(0)}\left(\frac{\partial \mathbf{v}_s}{\partial s}\otimes\mathbf{v}_s-K\mathbf{v}_s\otimes\mathbf{v}_n\right)+\boldsymbol{\mathscr{E}}^{(1)}_n\mathbf{v}_s\otimes\mathbf{v}_n+\boldsymbol{\mathscr{E}}^{(1)}_z\mathbf{v}_n\otimes\mathbf{v}_z-f\,\mathbf{C}^{(1)}\mathbf{v}_n\right\}\\ +\rho_0 g\hat{\mathbf{C}}^{(0)}\frac{\partial \boldsymbol{\xi}}{\partial s}+\mathbf{p}^{*(0)}_s-\mathbf{w}^{*(1)}_s-\mathbf{R}^{(1)}\mathbf{v}_s+\mathbf{J}_s = 0,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\rho_0\left\{\mathbf{C}^{(1)}\frac{\partial \mathbf{v}_n}{\partial t}+\boldsymbol{\mathscr{E}}^{(0)}\left(\frac{\partial \mathbf{v}_n}{\partial s}\otimes\mathbf{v}_s+K\mathbf{v}_s\otimes\mathbf{v}_s\right)+\boldsymbol{\mathscr{E}}^{(1)}_n\mathbf{v}_n\otimes\mathbf{v}_n+\boldsymbol{\mathscr{E}}^{(1)}_z\mathbf{v}_n\otimes\mathbf{v}_z+f\,\mathbf{C}^{(1)}\mathbf{v}_s\right\}\\ +\rho_0 g\hat{\mathbf{C}}^{(1)}_{\phi_n}\boldsymbol{\xi}+\mathbf{p}^{*(1)}_n-\mathbf{w}^{*(1)}_n+\mathbf{R}^{(1)}\mathbf{v}_n+\mathbf{J}_n = 0.\end{aligned} \tag{4.9}$$

In these expressions quantities written as bold script capital letters are third-order tensors in N dimensions; hence

$$(\boldsymbol{\mathscr{E}}\mathbf{a}\otimes\mathbf{b})_i = \mathscr{E}_{ijk}a_j b_k,$$

with $i,j,k = 1,2,\ldots,N$. Equation (4.8) corresponds to momentum balance in the s-direction, and (4.9) is that in the n-direction. An equation for the third component is incorporated in (4.8) and (4.9) as the hydrostatic-pressure assumption (2.10) has been used. The unknown field variables are $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$ and $\boldsymbol{\xi}$. The various indexed coefficients $\mathbf{C}$, $\mathbf{R}$ and $\mathscr{E}$ are known when the bathymetry is prescribed and when the shape functions are selected, all of which are defined in appendix A. The quantities carrying asterisks are the driving forces due to the wind and the atmospheric pressure gradient. These terms are also expressible as cross-sectional integrals, and are known when wind stress and atmospheric pressure gradients are prescribed. The terms in curly brackets are the accelerations in the longitudinal and transverse directions, those involving the $\mathscr{E}$'s are nonlinear and represent advection, and terms involving f account for the Coriolis effects. The first two terms in the second lines comprise all external forces, namely pressure gradient due to surface elevation, atmospheric pressure and wind, and together constitute the geostrophic balance. The term involving the matrix $\mathbf{R}^{(1)}$ accounts for bottom friction. It can consistently only be accounted for in a fluid that permits non-trivial viscous or turbulent stresses, of which the effect is collectively represented by $\mathbf{J}_s$ and $\mathbf{J}_n$. For completeness these quantities are defined in appendix A. For a particular closure model they must be given in terms of the independent field variables.

Equations (4.6), (4.8) and (4.9) do not yet form a complete determinate system of equations. A further equation is needed. This equation can again be deduced from (3.5) and the condition to arrive at this equation is $\delta I_5 = 0$, implying that

$$\begin{aligned}\int_{\partial\Omega_n} \frac{\operatorname{grad} F_H}{\|\operatorname{grad} F_H\|}\cdot \mathbf{v}\,\delta\lambda_3\,da &= \int_{\partial\Omega_n}\left(\frac{1}{J}\frac{\partial H}{\partial s}v_s + \frac{\partial H}{\partial n}v_n - v_z\right)\delta\lambda_3\frac{da}{l_H}\\
&= \int_{s_0}^{s_1} ds\left\{\int_{B^-}^{B^+}\frac{\partial H}{\partial s}v_s\,dn + \int_{B^-}^{B^+} J\frac{\partial H}{\partial n}v_n\,dn - \int_{B^-}^{B^+} Jv_z\,dn\right\}_{z=H}\delta\lambda_3\\
&= \int_{s_0}^{s_1} ds\,\delta\boldsymbol{\lambda}_3^{\mathrm{T}}\left\{\int_{B^-}^{B^+}\frac{\partial H}{\partial s}\boldsymbol{\psi}\otimes\boldsymbol{\phi}\,dn\,\mathbf{v}_s + \int_{B^-}^{B^+} J\frac{\partial H}{\partial n}\boldsymbol{\psi}\otimes\boldsymbol{\phi}\,dn\,\mathbf{v}_n\right.\\
&\quad\left. - \int_{B^-}^{B^+} J\boldsymbol{\psi}\otimes\boldsymbol{\phi}\,dn\,\mathbf{v}_z\right\}_{z=H} = 0,\end{aligned} \tag{4.10}$$

or in view of the definitions in appendix A,

$$\mathbf{H}_s^{(0)}\mathbf{v}_s + \mathbf{H}_n^{(1)}\mathbf{v}_n - \mathbf{H}^{(1)}\mathbf{v}_z = \mathbf{0}. \tag{4.11}$$

In the first line of (4.10), (2.4) is used together with the component form of the gradient operator and with the definition of l_H (2.8e). In the second line (B 1) of appendix B is employed to transform the integral over the bottom surface into an integral along the s-axis and the transverse direction, B^+ and B^- denoting the values of n at the positive and negative shorelines. Equation (4.11) is an algebraic equation relating $\mathbf{v}_s$, $\mathbf{v}_n$ and $\mathbf{v}_z$, and may be regarded as a prediction equation for $\mathbf{v}_z$ when $\mathbf{v}_s$ and $\mathbf{v}_n$ are prescribed.

To summarize, (4.6), (4.8), (4.9) and (4.11) comprise a system of four vectorial equations for the four unknown N-vectors $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$, $\boldsymbol{\xi}$. They are analogous to the basic equations (2.2), (2.3), (2.5), (2.6) and (2.8) but unlike these, they represent the global behaviour of the motion as a consequence of the smoothing or averaging process over the cross-sections of the lake achieved by expanding the variables in the weighted

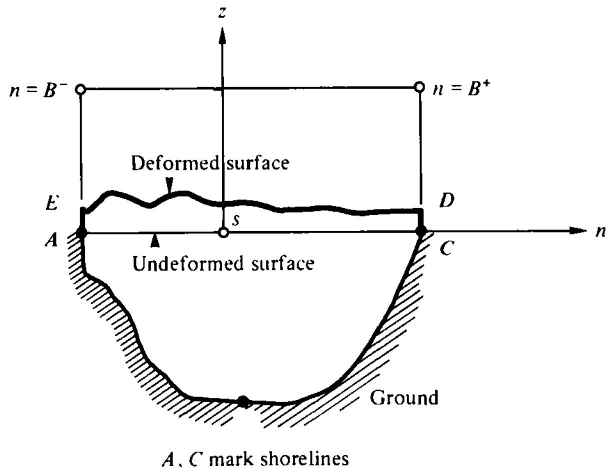

FIGURE 2. Idealized section.

residual expressions. Equation (4.6) is the analogue of the continuity equation in which the kinematic boundary conditions at the free and at the bottom surfaces are built in. Equation (4.11) is the global form of the bottom boundary condition, expressing the tangency of the flow at the bottom. The remaining two equations represent the two horizontal components of the momentum equations in which external driving forces are incorporated. It should finally be mentioned that with the definitions listed in appendix A the validity of the equations is restricted to small elevations of the free surface. In fact, it is assumed that shorelines do not change under motion. This amounts in figure 2 to the identification of the points A with E and C with D.

The spatially one-dimensional differential equations for the field variables $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$ and $\boldsymbol{\xi}$ must be complemented by closure conditions relating the macroscopic stress components with the variables $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$ and by boundary conditions. These are partly interrelated, because the closure condition determines the order of the differential equation. For an inviscid fluid model with vanishing viscous stress (4.6), (4.8), (4.9) and (4.11) are of first order in $\mathbf{v}_s$, $\mathbf{v}_n$ (and $\mathbf{v}_z$, which can be regarded as eliminated from (4.11)). Hence, using N shape functions for the variables, $3N$ boundary conditions must be prescribed. From (4.9) it is seen that the term involving $\partial \mathbf{v}_n/\partial s$ only arises in a formulation accounting for convective acceleration terms. Hence the number of boundary conditions depends on whether the term $\mathscr{E}^{(0)}\,\partial \mathbf{v}_n/\partial s \otimes \mathbf{v}_s$ is kept or not. If it is not, then $2N$ boundary conditions suffice; one may then require no flow through the end cross-sections. Hence the s-component of the physical velocity must vanish, which implies $\mathbf{v}_s = \mathbf{0}$ at $s = s_0$ and $s = s_e$, where s_0 and s_e signify the beginning and the end of the channel axis. When all connective terms are kept, N further conditions must be added.

Finally, to complete the initial-value problem, initial conditions for $\mathbf{v}_s$, $\mathbf{v}_n$, $\mathbf{v}_z$ and $\boldsymbol{\xi}$ must be prescribed.

4.2. *Cauchy-series expansion as an example for the shape function*

Explicit model equations hinge on the particular selection of the as-yet unspecified shape functions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. Numerical or physical considerations may guide us in the selection criteria. Simple powers, the so-called *Cauchy series expansions*, are such neutral representations, for which (index notation is now more appropriate)

$$\begin{aligned} v_s &= \sum_{k=0}^{N}\sum_{l=0}^{M} n^k z^l v_s^{(k,l)}(s,t), \quad v_z = \sum_{k=0}^{N}\sum_{k=0}^{M} n^k z^l z_z^{(k,l)}(s,t), \\ v_n &= \sum_{k=0}^{N}\sum_{k=0}^{M} n^k z^l v_n^{(k,l)}(s,t), \quad \xi = \sum_{k=0}^{N} n^k \xi^{(k)}(s,t). \end{aligned} \tag{4.12}$$

Such expansions though restricted to $M = N = 1$, are used in the theory of rods and jets (Green, Laws & Naghdi 1974; Green, Naghdi & Wenner 1974*a*, *b*; Green & Naghdi 1976; Naghdi 1978). For analytic fields, (4.12) are Taylor-series expansions about the lake axis, an interpretation which will be corroborated for problems treated in Raggio & Hutter (1982*a*), but under usual circumstances the functions $x^{(k,l)}$ are independent. For a Galerkin procedure the same expansions apply for the weighting functions.

The application of the principle of weighted residuals results in differential equations for the quantities that are indexed in (4.12) by the superscripts k and l. The global form of the continuity equation, bottom boundary condition and global momentum equations can be derived as indicated above; they read

$$Z^{(1)}_{(k+i,j)}\frac{\partial \xi^{(k)}}{\partial t} + \frac{\partial}{\partial s}[C^{(0)}_{(k+i,l+j)} v_s^{(k,l)}] - rC^{(1)}_{(k+i-1,l+j)} v_n^{(k,l)} - s\, C^{(1)}_{(k+i,l+j-1)} v_z^{(k,l)} = 0, \tag{4.13}$$

$$H^{(0)}_{s(k+i,l+j)} v_s^{(k,l)} + H^{(1)}_{n(k+i,l+j)} v_n^{(k,l)} - H^{(1)}_{(k+i,l+j)} v_z^{(k,l)} = 0, \tag{4.14}$$

$$\begin{aligned} &\rho_0\Big\{C^{(1)}_{(k+i,l+j)}\frac{\partial v_s^{(k,l)}}{\partial t} + C^{(0)}_{(k+p+i,l+q+j)}\left[\frac{\partial v_s^{(k,l)}}{\partial s} v_s^{(p,q)} - K v_s^{(k,l)} v_n^{(p,q)}\right] \\ &\quad + C^{(1)}_{(k+p+i,l+q+j)}[(k+1)\, v_s^{(k+1,l)} v_n^{(p,q)} + (l+1)\, v_s^{(k,l+1)} v_z^{(p,q)}] - f C^{(1)}_{(k+i,l+j)} v_n^{(k,l)}\Big\} \\ &\quad + \rho_0 g C^{(0)}_{(k+i,j)}\frac{\partial \xi^{(k)}}{\partial s} + p^{*(0)}_{s(i,j)} - w^{*(1)}_{s(i,j)} + R^{(1)}_{(k+i,l+j)} v_s^{(k,l)} + J^s_{(i,j)} = 0, \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\rho_0\Big\{C^{(1)}_{(k+i,l+j)}\frac{\partial v_n^{(k,l)}}{\partial t} + C^{(0)}_{(k+p+i,l+q+j)}\left[\frac{\partial v_n^{(k,l)}}{\partial s} v_s^{(p,q)} + K v_s^{(k,l)} v_s^{(p,q)}\right] \\ &\quad + C^{(1)}_{(k+p+i,l+q+j)}[(k+1)\, v_n^{(k+1,l)} v_n^{(p,q)} + (l+1)\, v_n^{(k,l+1)} v_z^{(p,q)}] + f C^{(1)}_{(k+i,l+j)} v_s^{(k,l)}\Big\} \\ &\quad + \rho_0 g (k+1)\, C^{(1)}_{(k+i,j)} \xi^{(k+1)} + p^{*(1)}_{n(i,j)} - w^{*(1)}_{n(i,j)} + R^{(1)}_{(k+i,l+j)} v_n^{(k,l)} + J^n_{(i,j)} = 0. \end{aligned} \tag{4.16}$$

In these equations r and s are free positive integers such that for every particular value of r and s an equation emerges. Furthermore, summation over doubly repeated superscripts is understood to extend in the first index from 0 to N and in the second index from 0 to M. As before the coefficients are cross-sectional integrals of geometrical quantities, terms carrying an asterisk are known external driving forces and the Js are due to internal friction. The definitions of these quantities in terms of cross-sectional integrals are given in appendix A. The balance law of mass agrees formally with the kinematic wave equation, and the basal boundary condition reduces, as before, to an algebraic statement relating the velocity components. The longitudinal [(4.15) and transverse (4.16)] momentum equations are written such that the various terms become readily identifiable. The first two lines involve the local, convective and Coriolis acceleration terms. In the third lines the external driving forces appear. These are pressure gradients (due to surface elevation and atmospheric pressure), wind forces and bottom friction.

5. Further simplifications – special models

Additional simplifications and preliminary calculations (presented in detail by Raggio & Hutter 1982*a, b*; Hutter & Raggio 1982) regarding gravitational waves in rotating basins of inviscid fluids are now introduced. Free oscillations of a zeroth-order model are shown to reduce to the Chrystal equation, thus elucidating the assumptions behind and the limitations of this classical equation. A first-order model with field variables expanded in terms of two shape functions is already general enough to predict waves correctly in rotating basins, for it replaces correctly the Kelvin wave-dynamics approach (discussed in § 1 and valid (at most) for the few lowest-order modes) and provides a channel model consistently derived from basic physical laws that reflects the properties of gravitational modes in rotating systems.

For the simplest model the expansions (3.2) are restricted to one constant term $\phi = \psi = 1$ representing a mean value over the cross-section. Governing equations are (4.6), (4.8), (4.9) and (4.11). However, the transverse mòmentum equation, (4.8), is not needed when longitudinal motions are considered, and the bottom boundary condition (4.11) is merely a prediction equation for the vertical velocity component. Hence, only (4.6) and (4.8) are relevant. Restricting attention to free motion (and therefore omitting all terms with an asterisk), ignoring Coriolis forces, convective acceleration and frictional resistence, the continuity equation (4.6) and the longitudinal momentum equation (4.8) reduce to

$$Z^{(1)}\frac{\partial \xi}{\partial t}+\frac{\partial}{\partial s}\left(C^{(0)}v_s\right) = 0, \quad C^{(1)}\frac{\partial v_s}{\partial t}+C^{(0)}g\frac{\partial \xi}{\partial s} = 0, \tag{5.1}$$

where vector notation has been dropped and the coefficients $Z^{(1)}$ and $C^{(j)}$ are defined as

$$C^{(0)} = \iint_Q dn\, dz = \text{cross-sectional area},$$

$$C^{(1)} = \iint_Q J\, dn\, dz = \text{cross-sectional area weighted with Jacobian},$$

$$Z^{(1)} = \int_{B^-}^{B^+} J\, dn = \text{channel width weighted with Jacobian}.$$

These contain the curvature of the lake axis (through J). The appearance of the Jacobian J of the curvilinear co-ordinate system as weighting factor of these integrals eliminates the element of subjectivity in selecting the channel axis that was mentioned in § 1. This is because different selections of the lake axis yield different values for these coefficients, thus automatically adjusting the equations accordingly.

If the channel axis is straight, $C^{(0)}$ and $C^{(1)}$ equal the cross-sectional area and $Z^{(1)}$ equals the width B of the channel, which may be functions of the coordinate s. This is exactly the situation analysed by Chrystal. Equations (5.1) therefore generalize the Chrystal equation to include curved basins. In fact, on introducing the transformations

$$\tilde{u} = C^{(0)}v_s, \quad \tilde{s} = \int_0^s Z^{(1)}(\xi)\, d\xi, \tag{5.2}$$

the equation (5.1) transform to

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = g\sigma(\tilde{s})\frac{\partial^2 \tilde{u}}{\partial \tilde{s}^2}, \quad \text{where} \quad \sigma(\tilde{s}) = \frac{[C^{(0)}]^2}{Z^{(1)}C^{(1)}}, \tag{5.3}$$

identical with Chrystal's original equation, in which the curvature was neglected. Its advantage is that the element of subjectivity in selecting the channel axis is eliminated, for the value of the coefficient $\sigma(s)$ is accordingly adjusted by the Jacobian. For further details see Raggio & Hutter (1982*b*).

The above analysis of the zeroth-order model brings out very clearly the conditions for which an equation with the same structure as the Chrystal equation is obtained: rotation is ignored and shape functions are independent of the cross-sectional co-ordinates.

A demonstration of the inclusion of the Chrystal equation as a very special case of the general model deduced in § 4 is a useful check on the equations. However, zeroth-order models are sometimes too crude, as they ignore variation of the velocities and surface elevation across the channel width. Extension to a first- or higher-order model allows incorporation of transverse variations of the field variables. When convective acceleration terms and frictional resistance are ignored the resulting equations are then appropriate for small-amplitude inviscid water motions in shallow, narrow, rotating basins, and solutions for simple geometries can be compared with those obtained from the classical (two-dimensional) tidal equations. This serves as a demonstration of the usefulness of the model. The variables in the tidal equations are independent of the vertical co-ordinate; this same independence is imposed in the channel model. In (4.12) we therefore set $M = 0$, $N = 1$ for all variables, choose $\boldsymbol{\phi} = \boldsymbol{\psi}$ (Galerkin) and call the emerging model first-order *in all variables*. Six variables occur, namely $\xi^{(0)}$, $v_s^{(0,0)}$, $v_n^{(0,0)}$, $\xi^{(1)}$, $v_s^{(1,0)}$ and $v_n^{(1,0)}$, and six equations can be deduced from (4.6), (4.8) and (4.9) ((4.11) being superfluous as a prediction equation for v_z). Omitting the second index in $v_s^{(1,0)}$ etc., as it plays no role, and introducing the vector

$$\mathbf{y} = (\xi^{(0)}, v_s^{(0)}, v_n^{(0)}, \xi^{(1)}, v_s^{(1)}, v_n^{(1)}), \tag{5.4}$$

the equations may be written as

$$\mathbf{A}\mathbf{y} = \mathbf{0}, \tag{5.5}$$

where

$$\mathbf{A}(\;) = \left[\begin{array}{ccc|ccc}
Z_{10}\dfrac{\partial(\;)}{\partial t} & \dfrac{\partial C_{00}(\;)}{\partial s} & 0 & Z_{11}\dfrac{\partial(\;)}{\partial t} & \dfrac{\partial C_{01}(\;)}{\partial s} & 0 \\
gC_{00}\dfrac{\partial(\;)}{\partial s} & C_{10}\dfrac{\partial(\;)}{\partial t} & -fC_{10}(\;) & gC_{01}\dfrac{\partial(\;)}{\partial s} & C_{11}\dfrac{\partial(\;)}{\partial t} & -fC_{11}(\;) \\
\hline
0 & fC_{10}(\;) & C_{10}\dfrac{\partial(\;)}{\partial t} & gC_{10}(\;) & fC_{11}(\;) & C_{11}\dfrac{\partial(\;)}{\partial t} \\
Z_{11}\dfrac{\partial(\;)}{\partial t} & \dfrac{\partial C_{01}(\;)}{\partial s} & -C_{10}(\;) & Z_{12}\dfrac{\partial(\;)}{\partial t} & \dfrac{\partial C_{02}(\;)}{\partial s} & -C_{11}(\;) \\
gC_{01}\dfrac{\partial(\;)}{\partial s} & C_{11}\dfrac{\partial(\;)}{\partial t} & -C_{11}(\;) & gC_{02}\dfrac{\partial(\;)}{\partial s} & C_{12}\dfrac{\partial(\;)}{\partial t} & -fC_{12}(\;) \\
0 & fC_{11}(\;) & C_{11}(\;) & gC_{11}(\;) & fC_{12}(\;) & C_{12}\dfrac{\partial(\;)}{\partial t}
\end{array}\right]. \tag{5.6}$$

In this matrix, in order to simplify notation we used C_{ij} for $C_{(j,0)}^{(i)}$ and Z_{ij} for $Z_{(j,0)}^{(i)}$. The first and fourth row are continuity statements, the second and fifth row describe momentum balance in the longitudinal direction and the third and sixth row express momentum balance in the transverse direction. The order of the rows and columns of the operator **A** have been selected such that as one deletes rows and columns from the lower right of **A**, other channel models which are zeroth-order in some and first-order in other variables, are obtained. Thus a hierarchy of models with decreasing complexity may be deduced. For instance the upper left 2×2 matrix corresponds to the zeroth-order model discussed above; the next extension to the 3×3 matrix would be a full zeroth-order model, but it is meaningless because there is no transverse pressure gradient produced by transverse variation of surface elevation, which does not occur (such a variation is necessary to obtain geostrophic balance). The model corresponding to the upper left 4×4 matrix is first-order in the surface elevation but zeroth-order in the velocities. This model would be the simplest version for which the effect of the rotating basin is reasonably accounted for. The remaining 5×5 and 6×6 models also include transverse variations in the velocity field. Clearly, in a model involving first-order terms the unidirectional zeroth-order motion is coupled with the remaining equations describing transverse variation of the field variables. A decoupling can only be achieved if all entries to the right of column two in rows one and two are zero. Non-dimensionalizing (5.5) with (5.6) by introducing appropriate scales shows that coupling results from, in general, three separate causes. One is due to the Coriolis parameter, the second is due to curvature along the channel axis and a third can be traced to 'asymmetry' of the cross-sections. Equations (5.5) and (5.6) have been specialized for a rectangular and a ring-shaped basin with constant depth by Raggio & Hutter (1982*a*) and Hutter & Raggio (1982).

For details of these calculations the reader is referred to these papers but as a matter of foresight we mention that (4.5) and (4.6) allow the prediction of oscillations with transverse structure, of Kelvin, Poincaré and inertial waves in rectangular basins of constant depth. Moreover, the reflection of a Kelvin wave propagating along one side of a half-open rectangular gulf and propagating in the opposite direction along the other side of the gulf (Taylor 1920) is approximately, but physically correctly and more transparently, predicted. That these phenomena, including those in polar co-ordinates (ring-shaped basins) which were studied with the full equations by Kelvin (1879); Lamb (1932) and Howard (1960), are correctly predicted by the equations is a good indication of their suitability, at least as far as first-class modes are concerned.

For the detailed presentation of this proof, including an application to a real basin, see Raggio & Hutter (1982*a*, *b*) and Hutter & Raggio (1982).

6. Summary and concluding remarks

For a thermally unstratified channel-like lake a spatially one-dimensional set of equations is derived that is suitable to describe the water motion in curved elongated enclosed basins. The equations are derived from the three-dimensional equations, formulated in a curvilinear co-ordinate system with one axis along the channel-like lake, by using the weighted residual technique, and the physically three-dimensional problem endowed with two-dimensional boundary conditions was transformed to a one-dimensional two-point boundary-value problem. Field variables were approximated by shape-function expansions, and the number of selected shape functions defined the order of the channel model.

The zeroth-order model (with one shape function) is a rational generalization of the Chrystal equation, in which the effects of the location and curvature of the lake axis are incorporated. This was not possible with the Chrystal theory, and thus an element of subjectivity in the selection of the lake axis is removed by our formulation. The first-order model and all higher-order models are appropriate for surface waves in *rotating* basins in that they reproduce qualitatively correctly the Kelvin and Poincaré wave structure and also allow the prediction of positively and negatively rotating amphidromic systems. These results are only discussed briefly as further details are to be published later (Raggio & Hutter 1982*a*, *b*; Raggio 1982; Hutter & Raggio 1982). They demonstrate, however, that this procedure for obtaining model equations for water motions in elongated basins is of value and might be pursued in other studies of slender body hydrodynamics. Free-surface rotating fluid-flow served as a demonstration 'that the method works'. An obvious extension would be the analysis of internal waves and corresponding large-amplitude thermocline motion. The reader who is familiar with the serious difficulties one encounters when trying to approximate the lake geometry in finite-difference studies of two-layer internal-wave studies in narrow elongated basins (see Bäuerle 1981) will undoubtedly realize that our approach is particularly suitable for such situations, since variations of the basin geometry are easily taken into account. To a certain extent this is also true for large amplitude internal waves which have been studied by Long (1956) and Benjamin (1966) for a fluid with constant and infinite depth. The resulting equations often conform with observations (see e.g. Osborne & Burch (1980), who use the Korteweg–de Vries (1895) equation), but preliminary examination of Lake of Zurich data (Mortimer 1980) suggests that large-amplitude internal waves in narrow basins may be bathymetry-dependent.

Further problems that might profit from a channel approach are for example an analysis including friction and turbulence effects to predict circulation patterns. Furthermore, a similar approach expanding field variables could be used in diffusion and advection problems of pollutants, or by expanding the field variables in the horizontal plane to describe seasonal processes, e.g. thermocline evolution.

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Appendix A

In this appendix we list the definitions of the cross-sectional coefficients, load and stress resultants that need be known in order that the dynamical equations of § 4 are completely known. Q denotes the cross-section of the lake, generally in its deformed configuration, but in our small-deflection approximation it may be chosen as the undeformed cross-sectional area. $H(s,n)$ denotes the depth function $z = H(s,n)$, and $n = B^{\pm}$ are the transverse co-ordinates of the shorelines (see figure 2). Commas denote partial derivatives, and shape functions carrying a hat are functions of n only.

Cross-sectional coefficients

$$\left.\begin{aligned}
&\mathbf{C}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn\, dz, && \mathbf{H}_n^{(m)} = \int_{B^-}^{B^+} J^m H_{,n} \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn \\
& && \qquad (z = H(s,n)),\\
&\mathbf{C}_{\phi_j}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi}_{,j}\, dn\, dz \quad (j = n, z), && \mathbf{Z}^{(m)} = \int_{B^-}^{B^+} J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn \\
& && \qquad (z = \xi(s,n,t)),\\
&\mathbf{C}_{\psi_j}^{(m)} = \iint_Q J^m \boldsymbol{\psi}_{,j} \otimes \boldsymbol{\phi}\, dn\, dz \quad (j = n, z), && \hat{\mathbf{Z}}^{(m)} = \int_{B^-}^{B^+} J^m \boldsymbol{\psi} \otimes \hat{\boldsymbol{\phi}}\, dn\, dz \\
& && \qquad (z = \xi(s,n,t)),\\
&\hat{\mathbf{C}}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \hat{\boldsymbol{\phi}}\, dn\, dz, && \mathscr{E}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi} \otimes \boldsymbol{\phi}\, dn\, dz,\\
&\hat{\mathbf{C}}_{\phi_j}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \hat{\boldsymbol{\phi}}_{,j}\, dn\, dz \quad (j = n, z), && \mathscr{E}_{,j}^{(m)} = \iint_Q J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi}_{,j} \otimes \boldsymbol{\phi}\, dn\, dz \\
& && \qquad (j = n, z),\\
&\mathbf{H}^{(m)} = \int_{B^-}^{B^+} J^m \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn \quad (z = H(s,n)), && \hat{\mathscr{E}}^{(m)} = \int_{B^-}^{B^+} J^m \boldsymbol{\psi} \otimes \hat{\boldsymbol{\phi}} \otimes \boldsymbol{\phi}\, dn\, dz \\
& && \qquad (z = \xi(s,n,t)),\\
&\mathbf{H}_s^{(m)} = \int_{B^-}^{B^+} J^m H_{,s} \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, dn \quad (z = H(s,n)), && \hat{\mathscr{E}}_{,j}^{(m)} = \int_{B^-}^{B^+} J^m \boldsymbol{\psi} \otimes \hat{\boldsymbol{\phi}}_{,j} \otimes \boldsymbol{\phi}\, dn\, dz, \\
& && \qquad (z = \xi(s,n,t),\ j = n, z).
\end{aligned}\right\} \tag{A 1}$$

These quantities are purely geometric. Analogously, the *macroscopic bottom-friction coefficient* is mainly a function of the geometry of the basin:

$$\mathbf{R}^{(m)} = \int_{B^-}^{B^+} \frac{J^m}{R} \boldsymbol{\psi} \otimes \boldsymbol{\phi}\, l_H(s,n)\, dn \quad (z = H(s,n)), \tag{A 2}$$

where $l_H(s,n)$ is defined in (2.8).

Load resultants

Let t_α^* $(\alpha = s, n)$ be the wind stress components and p_{at}^* the atmospheric pressure. We then define the *macroscopic wind load* and *macroscopic atmospheric pressurc gradients* as follows:

$$\left.\begin{aligned}
\mathbf{w}_\alpha^{*(m)} &= \int_{B^-}^{B^+} J^m \boldsymbol{\psi} t_\alpha^*\, dn \quad (z = \xi(s,n,t),\ \alpha = s, n),\\
p_\alpha^{*(m)} &= \iint_Q J^m \boldsymbol{\psi} \frac{\partial p_{\mathrm{at}}^*}{\partial \alpha}\, \mathrm{d}n\, \mathrm{d}z \quad (\alpha = s, n).
\end{aligned}\right\} \tag{A 3}$$

Stress resultants

These arise only when internal friction is accounted for. The two quantities $\mathbf{J}_s$, $\mathbf{J}_n$, arising in equations (4.8) and (4.9) have the form

$$\left.\begin{aligned} \mathbf{J}_s &= -\mathbf{p}_s^{\mathrm{T}} - \frac{\partial T_{11}^{(0)}}{\partial s} + \mathbf{T}_{11s}^{(0)} + \mathbf{T}_{11s}^{(-1)} + \mathbf{T}_{12n}^{(1)} + \mathbf{T}_{13z}^{(1)} + \mathbf{K}\mathbf{T}_{12}^{(0)} = 0, \\ \mathbf{J}_n &= -\mathbf{p}_n^{\mathrm{T}} - \frac{\partial \mathbf{T}_{12}^{(0)}}{\partial s} + \mathbf{T}_{22n}^{(1)} + \mathbf{T}_{23z}^{(1)} + K(\mathbf{T}_{11}^{(2)} - 2\mathbf{T}_{22}^{(0)}) = 0, \end{aligned}\right\} \tag{A 4}$$

and the stress resultants are of two types, namely the *macroscopic turbulent surface pressure*

$$\left.\begin{aligned} \mathbf{p}_s^{\mathrm{T}} &= \int_{B^-}^{B^+} \psi(T_{ij}^{\mathrm{E}} n_i n_j) \frac{\partial H}{\partial s} dn, \\ \mathbf{p}_n^{\mathrm{T}} &= \int_{B^-}^{B^+} J\psi(T_{ij}^{\mathrm{E}} n_i n_j) \left[1 + \left(\frac{\partial H}{\partial n}\right)^2\right]^{\frac{1}{2}} dn, \end{aligned}\right\} \quad (z = H(s,n)), \tag{A 5}$$

and the macroscopic stress resultants

$$\mathbf{T}_{ij}^{(m)} = \iint_Q J^m \psi T_{ij}^{\mathrm{E}}\, dn\, dz, \quad \mathbf{T}_{ij\alpha}^{(m)} = \iint_Q J^m \psi_{,\alpha} T_{ij}^{\mathrm{E}}\, dn\, dz \quad (\alpha = n, z),$$
$$T_{ijs}^{(m)} = \iint_Q \frac{\partial J}{\partial s} J^m \psi T_{ij}^{\mathrm{E}}\, dn\, dz. \tag{A 6}$$

When *Cauchy-series expansions* (4.13) are used there are fewer *cross-sectional coefficients*, namely

$$\left.\begin{aligned} C_{(i,j)}^{(m)} &= \iint_Q J^m n^i z^j\, dn\, dz, \quad H_{\alpha(i,j)}^{(m)} = \int_{B^-}^{B^+} J^m n^i H^j(s,n) \frac{\partial H(s,n)}{\partial \alpha} \quad (\alpha = s,n), \\ H_{(i,j)}^{(m)} &= \int_{B^-}^{B^+} J^m n^i H^j(s,n)\, dn, \\ Z_{(i,j)}^{(m)} &= \int_{B^-}^{B^+} J^m n^i \xi^j(s,n,t)\, dn, \quad R_{(i,j)}^{(m)} = \int_{B^-}^{B^+} \frac{J^m}{R} n^i H^j(s,n)\, l_H(s,n)\, dn. \end{aligned}\right\} \tag{A 7}$$

The atmospheric pressure and wind-stress terms are

$$\left.\begin{aligned} w_{\alpha(i,j)}^{*(m)} &= \int_{B^-}^{B^+} J^m n^i \xi^j(s,n,t)\, t_\alpha^*\, dn, \\ p_{\alpha(i,j)}^{*(m)} &= \iint_Q J^m n^i z^j \frac{\partial p_{\mathrm{at}}^*}{\partial \alpha}\, dn\, dz \end{aligned}\right\} \quad (\alpha = s,n), \tag{A 8}$$

and the turbulent stress terms $J_{(i,j)}$ are

$$\left.\begin{aligned} J_{(i,j)}^s &= -p_{s(i,j)}^{\mathrm{T}} - \frac{\partial T_{11(i,j)}^{(0)}}{\partial s} + K'KT_{11(i+2,j)}^{(-1)} + iT_{12(i-1,j)}^{(1)} - KT_{12(i,j)}^{(0)} + jT_{13(i,j-1)}^{(1)}, \\ J_{(i,j)}^n &= -p_{n(i,j)}^{\mathrm{T}} - \frac{\partial T_{12(i,j)}^{(0)}}{\partial s} + iT_{22(i,j)}^{(1)} + jT_{23(i,j)}^{(1)} + K(T_{11(i,j)}^{(2)} - 2T_{22(i,j)}^{(0)}), \end{aligned}\right\} \tag{A 9}$$

with

$$\left.\begin{aligned} p^{\mathrm{T}}_{s(i,j)} &= \int_{B^-}^{B^+} n^i H^j(s,n)\frac{\partial H(s,n)}{\partial s}(T^{\mathrm{E}}_{kl}n_k n_l)\,dn, \\ p^{\mathrm{T}}_{n(i,j)} &= \int_{B^-}^{B^+} Jn^i H^j(s,n)\left[1+\left(\frac{\partial H}{\partial n}\right)^2\right]^{\frac{1}{2}}(T^{\mathrm{E}}_{kl}n_k n_l)\,dn, \\ T^{(m)}_{kl(i,j)} &= \iint_Q J^m n^i z^j T_{kl}\,dn\,dz, \end{aligned}\right\} \quad \begin{aligned} &(k,l = 1,2,3) \\ &(i = 1,2,\ldots,N). \\ &(j = 1,2,\ldots,M) \end{aligned} \qquad \text{(A 10)}$$

Appendix B

Global continuity equation with incorporated kinematic boundary conditions

In this appendix we derive the global form of the continuity equation which expresses mass balance in a one-dimensional form. To this end the following preliminary calculations are needed.

Let $F(s,n,z,t) \equiv 0$ be the equation defining the boundary $\partial\Omega$ of the domain Ω (see (2.4)), and let $d\mathbf{a}$ be the surface element perpendicular to $\partial\Omega$. Elementary vector calculus and differential geometry then shows that in the (s,n,z)-co-ordinate system

$$\left.\begin{aligned} d\mathbf{a} &= \left(-\frac{\partial F}{\partial s}, -J\frac{\partial F}{\partial n}, J\right)dn\,ds = d\mathbf{a}_s + d\mathbf{a}_l, \\ d\mathbf{a}_s &= \left(-\frac{\partial F}{\partial s}, 0, 0\right)dn\,ds = -(1,0,0)\,da'_s\,ds = \mathbf{n}_s.da_s, \\ d\mathbf{a}_l &= \left(0, -\frac{\partial F}{\partial n}, 1\right)J\,dn\,ds = (0, n_{l_2}, n_{l_3})\,da'_l J\,ds = \mathbf{n}_l da_l, \\ da &= \|d\mathbf{a}\| = Jl\,dn\,ds = da'\,ds, \\ l &\equiv \left[1+\left(\frac{\partial F}{\partial n}\right)^2 + \frac{1}{J^2}\left(\frac{\partial F}{\partial s}\right)^2\right]^{\frac{1}{2}} = \|\operatorname{grad} F\|. \end{aligned}\right\} \qquad \text{(B 1)}$$

Here da_s is the algebraic surface element of the projection of the mantle element $d\mathbf{a}$ into the cross-sectional plane (see figure 3), $\mathbf{n}_s$ and $\mathbf{n}_l$ are unit vectors along the negative s-direction and perpendicular to the cross-sectional periphery but within the cross-sectional plane. Thus $da_l = da'_l J\,ds$, with $da'_l = (1+(\partial F/\partial n)^2)^{\frac{1}{2}}$, is the algebraic area element projected on a cylindrical surface that is parallel to s and sweeps out the periphery of the cross-section. Also, depending on the choice of the surfaces (2.4), we write l as l_H and $l_\xi = 1 + O(\xi^2)$, respectively, consistently ignoring $O(\xi^2)$ terms. This unimportant approximation has been introduced as a matter of convenience, it implies that the boundary changes AE and CD in figure 2 are ignored.

We also need the following formulae:

$$\left.\begin{aligned} &\frac{\partial}{\partial s}\iint_Q f\,dn\,dz = \iint_Q \frac{\partial f}{\partial s}dn\,dz + \oint_{\partial Q} f\,da'_s, \\ &\iint_Q \frac{\partial f}{\partial n}dn\,dz = \oint_{\partial Q} f n_{l_2} da'_l, \quad \iint_Q \frac{\partial f}{\partial z}dn\,dz = \oint_{\partial Q} f n_{l_3} da'_l. \end{aligned}\right\} \qquad \text{(B 2)}$$

Q denotes the cross-section and ∂Q its periphery.

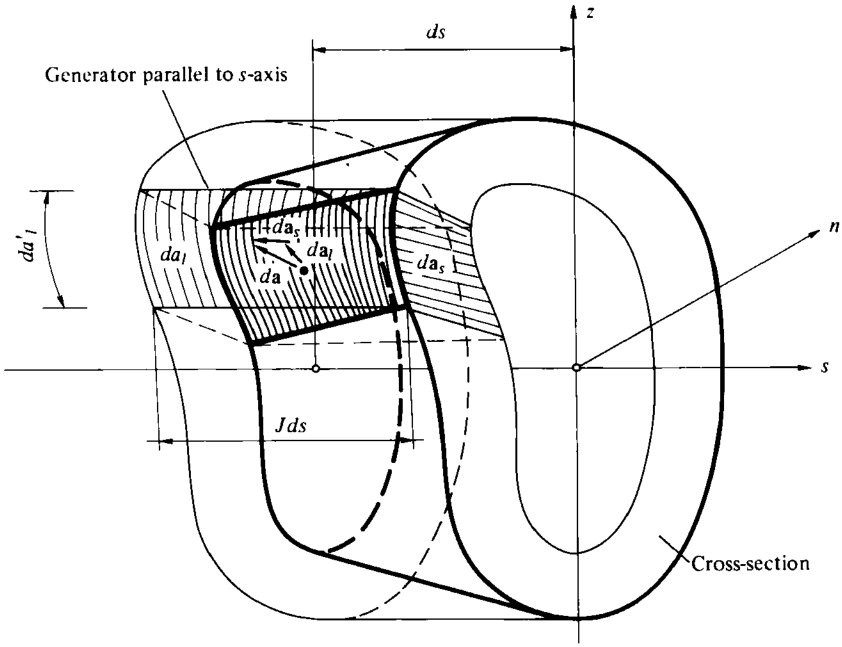

FIGURE 3. Channel-like element of length ds with surface element da on its mantle surface. Projections of this element into the cross-section and onto the cylindrical surface are shown by light dashed areas.

With these preliminary calculations we are now in the position to derive a weak statement of the mass-balance equation. We start by writing δI_1 in (3.5) as

$$\iiint_\Omega \operatorname{div} \mathbf{v}\, \delta\lambda\, dv = \int_{s_0}^{s} \delta\boldsymbol{\lambda}^{\mathrm{T}} \left\{ \iint_Q \boldsymbol{\psi} \left(\frac{1}{J}\frac{\partial v_s}{\partial s} + \frac{\partial v_n}{\partial n} + \frac{\partial v_z}{\partial z} - \frac{K}{J} v_n \right) J\, dn\, dz \right\} ds$$

$$= \int_{s_0}^{s} \delta\boldsymbol{\lambda} . \left\{ \iint_Q \boldsymbol{\psi} \frac{\partial v_s}{\partial s} dn\, dz + \iint_Q \boldsymbol{\psi} \frac{\partial v_n}{\partial_n} J\, dn\, dz + \iint_Q \boldsymbol{\psi} \frac{\partial v_z}{\partial z} J\, dn\, dz - K \iint_Q \boldsymbol{\psi} v_n\, dn\, dz \right\} ds \tag{B 3}$$

The first three innermost cross-sectional integrals can be transformed with the aid of (B 2) so that (B 3) may be written as

$$\int_{s_0}^{s} \delta\boldsymbol{\lambda} . \left\{ \left[\frac{\partial}{\partial s} \left(\iint_Q \boldsymbol{\psi} v_s\, dn\, dz \right) - \oint_{\partial Q} \boldsymbol{\psi} v_s\, da_s' \right] \right.$$
$$+ \left[\oint_{\partial Q} \boldsymbol{\psi} J v_n n_{l_2} da_l' - \iint_Q \frac{\partial \boldsymbol{\psi}}{\partial n} J v_n\, dn\, dz + K \iint_Q \boldsymbol{\psi} v_n\, dn\, dz \right]$$
$$\left. + \left[\oint_{\partial Q} \boldsymbol{\psi} J v_z n_{l_3} da_l' - \iint_Q \frac{\partial \boldsymbol{\psi}}{\partial z} J v_z\, dn\, dz \right] - K \iint_Q \boldsymbol{\psi} v_n\, dn\, dz \right\} ds; \tag{B 4}$$

or, after rearranging terms,

$$\iiint_\Omega \operatorname{div} \mathbf{v}\, \delta\lambda\, dv = \int_{s_0}^{s} \delta\boldsymbol{\lambda} . \left\{ - \oint_{\partial Q} \boldsymbol{\psi} v_s\, da_s' + \oint_{\partial Q} \boldsymbol{\psi} J v_\beta n_{l\beta}\, da_l' + \frac{\partial}{\partial s} \left(\iint_Q \boldsymbol{\psi} v_s\, dn\, dz \right) \right.$$
$$\left. - \iint_Q \left(\frac{\partial \boldsymbol{\psi}}{\partial n} J v_n - \frac{\partial \boldsymbol{\psi}}{\partial z} J v_z \right) dn\, dz \right\} ds \quad (\beta = n, z), \tag{B 5}$$

in which summation over Greek indices is understood. To eliminate the contour integrals, consider the weighted residual expresses δI_3 and δI_5 for the kinematic boundary conditions (3.6c, e). Set $\delta\lambda_3 = \delta\lambda_2$, which is permissible, and form $\delta I_3 + \delta I_5 = 0$; this yields

$$\iint_{\partial\Omega_\sigma} \frac{1}{|\operatorname{grad} F\xi|}\frac{dF_\xi}{dt}\delta\lambda_2\,da + \iint_{\partial\Omega_*}\frac{1}{|\operatorname{grad} F_H|}\frac{dF_H}{dt}\delta\lambda_3\,da = \oiint_{\partial\Omega}\frac{1}{|\operatorname{grad} F|}\frac{dF}{dt}\delta\lambda_2\,da$$
$$= \oiint_{\partial\Omega}\frac{1}{|\operatorname{grad} F|}\frac{\partial F}{\partial t}\delta\lambda_2\,da + \oiint_{\partial\Omega}\mathbf{v}\,.\,\frac{\operatorname{grad} F}{|\operatorname{grad} F|}\delta\lambda_2\,da = 0.$$
$$= \oiint_{\partial\Omega}|\operatorname{grad} F|^{-1}\frac{\partial F}{\partial t}\delta\lambda_2\,da - \oiint_{\partial\Omega}\mathbf{v}\,\delta\lambda_2\,d\mathbf{a} = 0. \tag{B 6}$$

The last identity follows, since grad $F/\|\operatorname{grad} F\|$ is the inward unit normal (see (2.4)). Splitting the surface element $d\mathbf{a}$ into the two perpendicular surface elements $d\mathbf{a}_s$ and $d\mathbf{a}_l$ gives

$$\oiint_{\partial\Omega}|\operatorname{grad} F|^{-1}\frac{\partial F}{\partial t}\delta\lambda_2\,da - \oiint_{\partial\Omega}\mathbf{v}\,.\,\delta\lambda_2\,d\mathbf{a}_s - \oiint_{\partial\Omega}\mathbf{v}\,.\,\delta\lambda_2\,d\mathbf{a}_l$$
$$= \oiint_{\partial\Omega}|\operatorname{grad} F|^{-1}\frac{\partial F}{\partial t}\delta\lambda_2\,da - \oiint_{\partial\Omega}\mathbf{v}\,.\,\mathbf{n}_s\,\delta\lambda_2\,da_s - \oiint_{\partial\Omega}\mathbf{v}\,.\,\mathbf{n}_l\,\delta\lambda_2\,da_l$$
$$= \int_{s_0}^{s}\left\{\oint_{\partial\Omega}|\operatorname{grad} F|^{-1}\frac{\partial F}{\partial t}\delta\lambda_2\,da' + \oint_{\partial\Omega}v_s\,\delta\lambda_2\,da_s' - \oint_{\partial\Omega}v_\beta n_{l\beta}\,\delta\lambda J\,da_l'\right\}ds;$$

or, after expansion of the weighting function $\delta\lambda_2$,

$$\oiint_{\partial\Omega}|\operatorname{grad} F|^{-1}\frac{dF}{dt}\delta\lambda_2\,da$$
$$= \int_{s_0}^{s}\delta\boldsymbol{\lambda}_2^{\mathrm{T}}\,.\left\{\oint_{\partial\Omega}\boldsymbol{\psi}\,|\operatorname{grad} F|^{-1}\frac{\partial F}{\partial t}da' + \oint_{\partial\Omega}\boldsymbol{\psi} v_s\,da_s' - \oint_{\partial\Omega}\boldsymbol{\psi} J v_\beta n_{l\beta}\,da_l'\right\}ds. \tag{B 7}$$

When the expressions (B 5) and (B 7) are added together it is seen that the contour integrals cancel, so that

$$\delta I_1 + \delta I_3 + \delta I_5 = 0$$

implies

$$\int_{s_0}^{s}\delta\boldsymbol{\lambda}\,.\left\{\int_{B^-}^{B^+}\boldsymbol{\psi}\otimes\boldsymbol{\phi}^{\mathrm{T}}J\,dn\frac{\partial\xi}{\partial t} + \frac{\partial}{\partial s}\left(\iint_Q\boldsymbol{\psi}\otimes\boldsymbol{\phi}^{\mathrm{T}}\,\mathrm{dn}\,dz\,\mathbf{v}_s\right)\right.$$
$$\left. - \iint_Q\frac{\partial\boldsymbol{\psi}}{\partial n}\otimes\boldsymbol{\phi}^{\mathrm{T}}J\,dn\,dz\,\mathbf{v}_n - \iint_Q\frac{\partial\boldsymbol{\psi}}{\partial z}\otimes\boldsymbol{\phi}^{\mathrm{T}}J\,dn\,dz\,\mathbf{v}_z\right\}ds = 0,$$

or since $\delta\boldsymbol{\lambda}$ is arbitrary, and in view of the definitions in appendix A

$$\hat{\mathbf{Z}}^{(1)}\frac{\partial\xi}{\partial t} + \frac{\partial}{\partial s}(\mathbf{C}^{(0)}\mathbf{v}_s) - \mathbf{C}^{(1)}_{\psi n}\mathbf{v}_n - \mathbf{C}^{(1)}_{\psi z}\mathbf{v}_z = 0. \tag{B 8}$$

This is the continuity equation we were looking for. Equation (B 8) is given in the main text as equation (4.6).

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